in which

$$\chi_k{}^\mu(\hat{r})=\sum_{\tau=\pm\frac{1}{2}} C[l(k),\tfrac{1}{2},j;\mu-\tau,\tau]Y_{l(k)}{}^{\mu-\tau}(\hat{r})\chi^\tau,$$

$$g_k(E,r)=[p(E+1)/\pi]^{\frac{1}{2}}\{\ \}_+,$$

$$f_k(E,r)=i[p(E-1)/\pi]^{\frac{1}{2}}\{\ \}_-,$$

$$\{\ \}_\pm=[e^{\nu\pi/2}|\Gamma(|k|+i\nu)|\,|x|^{|k|-1}/\Gamma(2|k|+1)]\times\{(|k|+i\nu)e^{i\eta}e^{-x/2}\times{}_1F_1(|k|+1+i\nu,2|k|+1,x)\pm\text{c.c.}\},$$

$$x=-2ipr.$$

By means of the methods described in Ref. 6, the scattering solution given by Eq. (3) may be written in the Johnson-Deck form

$$\psi=\{N+i\lambda M\gamma_5\boldsymbol{\sigma}\cdot(\hat{p}-\hat{r})+L[\boldsymbol{\sigma}\cdot(\hat{p}-\hat{r})](\boldsymbol{\sigma}\cdot\hat{p})\}U(\hat{p}),\quad(8)$$

and consequently all the results[4] pertaining to this form may be used. Here, $U(\hat{p})$ is a plane wave spinor of arbitrary polarization. The functions N, M, and L are given by

$$N=2\sum_{k=1}^{\infty}(-1)^k x^{k-1}e^{x/2}e^{\nu\pi/2}\times[\Gamma(k-i\nu)/\Gamma(2k+1)](k^2+\lambda^2)^{\frac{1}{2}}\times{}_1F_1(k-i\nu,2k+1,x)[P_{k-1}'(\hat{p}\cdot\hat{r})-P_k'(\hat{p}\cdot\hat{r})],\quad(9)$$

$$M=-\tfrac{1}{2}\Gamma(1-i\nu)e^{\nu\pi/2}e^{i\mathbf{p}\cdot\mathbf{r}}{}_1F_1[1+i\nu,2,i(pr-\mathbf{p}\cdot\mathbf{r})],\quad(10)$$

$$L=\tfrac{1}{2}(N_{\lambda=0}-N).\quad(11)$$

For $\lambda=0$, the series for N can be summed to yield

$$N_{\lambda=0}=\Gamma(1-i\nu)e^{\nu\pi/2}e^{i\mathbf{p}\cdot\mathbf{r}}{}_1F_1[i\nu,1,i(pr-\mathbf{p}\cdot\mathbf{r})].\quad(12)$$

In Eq. (9) the prime indicates the derivative of the Legendre polynomial P with respect to its argument $(\hat{p}\cdot\hat{r})$.

These functions may now be directly compared with the corresponding ones for the Dirac-Coulomb case[6] for which all three functions are given as infinite series. The main difference is that the scattering solution for the Biedenharn Hamiltonian has simpler parameters in the ${}_1F_1$ functions and the function M is expressible in closed form. In fact, M is exactly the same as that for the Sommerfeld-Maue approximation, while N and L differ from the Sommerfeld-Maue approximation by order λ^2.

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Theory of Unstable Particles

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A new method in the theory of unstable particles is introduced. It is applied in this paper to a simple model to show how the exponential regime of the decay can be isolated systematically. It is further shown that our method prescribes the precise conditions to which the initial wave function must be submitted for this exponential decay to ensue. The prescription of these conditions constitutes in fact a definition of an "unstable particle" in quantum theory.

INTRODUCTION

In this paper we shall be concerned with the classification of the decay regimes of unstable particles. The first task of the theory is that of isolating the exponential decay. It will be shown, by means of an illustrative model, how this can be accomplished systematically. Furthermore, it will be shown that the method is sufficiently powerful to allow for the determination of the precise conditions to be imposed on the initial wave packet for such an exponential decay.

The mathematical apparatus has been introduced recently by one of us[1] and applied to nonequilibrium statistical mechanics. Analogies with the results of Ref. 1 are numerous and will occasionally be noted. Previous treatments of unstable states are surveyed in Ref. 2.

For illustration we shall carry out our calculations with a model due to Wigner and Weisskopf.[3,4]

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[1] G. Sandri, Ann. Phys. (N. Y.) **24**, 332 (1963). This paper contains the lectures on the Foundations of Non-Equilibrium Statistical Mechanics, given at Rutgers (1961–62).
[2] M. Goldberger and K. Watson, *Collision Theory* (John Wiley & Sons, Inc., New York, 1964).
[3] E. Wigner and V. Weisskopf, Z. Physik **63**, 62 (1930).
[4] M. Wellner, Phys. Rev. **118**, 875 (1960).

The equations for the model are written as

$$i(\partial\theta/\partial t)+\nabla^2\theta=\epsilon U\chi\,, \tag{1}$$

$$i(\partial\chi/\partial t)-\mu\chi=\epsilon\int U\theta d\mathbf{x}\,, \tag{2}$$

where θ is a function of $\mathbf{x}$ and t while χ is a function of t alone. $U(\mathbf{x})$ is a (given) real form factor. μ is real. The model can be made to correspond to the process $\Lambda \leftrightarrows N+\theta$ by taking χ to be the wave function of the Λ particle and $\theta(\mathbf{x},t)$ to be the wave function of the $N-\theta$ system.[5] The criterion for stability of the eigenstate corresponding to the "physical Λ" is exhibited in Ref. 4.

We shall show that the quantity

$$P\equiv\chi^*\chi \tag{3}$$

exhibits an exponential decay under precise conditions calculable from (1) and (2). The behavior of the flux

$$i\int(\theta^*\nabla\theta-\theta\nabla\theta^*)\cdot d\mathbf{s}$$

is analogous.

EXPONENTIAL REGIME

We obtain from (2) and (3)

$$i(\partial P/\partial t)=\epsilon 2i\,\mathrm{Im}\left[\chi^*\int U\theta d\mathbf{x}\right]. \tag{4}$$

Perturbation expansion in powers of ϵ leads to terms that diverge for large time. Such an expansion is therefore inadequate to describe decay. In order to obtain an expansion in ϵ which is valid for long times, and which will therefore enable us to follow the decay of P, we introduce "extended functions,"[1] as follows. We embed the temporal domain of θ, χ, and P into a space of three independent time variables τ_0, τ_1, and τ_2 and we consider "extended" functions $\boldsymbol{\theta}(\mathbf{x},\tau_0,\tau_1,\tau_2)$, $\boldsymbol{\chi}(\tau_0,\tau_1,\tau_2)$ and $\mathbf{P}(\tau_0,\tau_1,\tau_2)$ which have the following three properties:

(i) The extended functions coincide with θ, χ, and P along the "trajectories"

$$\tau_0=t\,,\quad \tau_1=\epsilon t\,,\quad \tau_2=\epsilon^2 t. \tag{5}$$

This is expressed as

$$\boldsymbol{\theta}(\mathbf{x},t,\epsilon t,\epsilon^2 t)=\theta(\mathbf{x},t)\,, \tag{6}$$

$$\boldsymbol{\chi}(t,\epsilon t,\epsilon^2 t)=\chi(t)\,, \tag{7}$$

$$\mathbf{P}(t,\epsilon t,\epsilon^2 t)=P(t)\,. \tag{8}$$

(ii) The time derivatives of $\boldsymbol{\theta}$, $\boldsymbol{\chi}$, and $\mathbf{P}$ are given by

$$\frac{\partial\boldsymbol{\theta}}{\partial t}=\frac{\partial\boldsymbol{\theta}}{\partial\tau_0}+\epsilon\frac{\partial\boldsymbol{\theta}}{\partial\tau_1}+\epsilon^2\frac{\partial\boldsymbol{\theta}}{\partial\tau_2}\,, \tag{9}$$

$$\frac{\partial\boldsymbol{\chi}}{\partial t}=\frac{\partial\boldsymbol{\chi}}{\partial\tau_0}+\epsilon\frac{\partial\boldsymbol{\chi}}{\partial\tau_1}+\epsilon^2\frac{\partial\boldsymbol{\chi}}{\partial\tau_2}\,, \tag{10}$$

$$\frac{\partial\mathbf{P}}{\partial t}=\frac{\partial\mathbf{P}}{\partial\tau_0}+\epsilon\frac{\partial\mathbf{P}}{\partial\tau_1}+\epsilon^2\frac{\partial\mathbf{P}}{\partial\tau_2}\,. \tag{11}$$

(iii) $\boldsymbol{\theta}$, $\boldsymbol{\chi}$, and $\mathbf{P}$ have asymptotic expansions in ϵ which are uniformly valid in the three independent variables τ_0, τ_1, τ_2 and which have the form

$$\boldsymbol{\theta}=\boldsymbol{\theta}_0+\epsilon\boldsymbol{\theta}_1+\epsilon^2\boldsymbol{\theta}_2+O(\epsilon^3)\,, \tag{12}$$

$$\boldsymbol{\chi}=\boldsymbol{\chi}_0+\epsilon\boldsymbol{\chi}_1+\epsilon^2\boldsymbol{\chi}_2+O(\epsilon^3)\,, \tag{13}$$

$$\mathbf{P}=\mathbf{P}_0+\epsilon\mathbf{P}_1+\epsilon^2\mathbf{P}_2+O(\epsilon^3)\,. \tag{14}$$

Substituting the expansions (9) to (14) into (1), (2), and (4) we find, in zeroth order

$$i(\partial\boldsymbol{\theta}_0/\partial\tau_0)+\nabla^2\boldsymbol{\theta}_0=0\,, \tag{15}$$

$$i(\partial\boldsymbol{\chi}_0/\partial\tau_0)-\mu\boldsymbol{\chi}_0=0\,, \tag{16}$$

$$i(\partial\mathbf{P}_0/\partial\tau_0)=0\,. \tag{17}$$

In first order we obtain from (1), (2), and (4), respectively,

$$i\frac{\partial\boldsymbol{\theta}_0}{\partial\tau_1}+i\frac{\partial\boldsymbol{\theta}_1}{\partial\tau_0}+\nabla^2\boldsymbol{\theta}_1=U\boldsymbol{\chi}_0 \tag{18}$$

$$i\frac{\partial\boldsymbol{\chi}_0}{\partial\tau_1}+i\frac{\partial\boldsymbol{\chi}_1}{\partial\tau_0}-\mu\boldsymbol{\chi}_1=\int U\boldsymbol{\theta}_0 d\mathbf{x} \tag{19}$$

$$i\frac{\partial\mathbf{P}_0}{\partial\tau_1}+i\frac{\partial\mathbf{P}_1}{\partial\tau_0}=2i\,\mathrm{Im}\left[\boldsymbol{\chi}_0{}^*\int U\boldsymbol{\theta}_0 d\mathbf{x}\right]. \tag{20}$$

The solution of (15) will be written as

$$\boldsymbol{\theta}_0(\tau_0)=e^{+i\nabla^2\tau_0}\boldsymbol{\theta}_0(0)\,. \tag{21}$$

The "simple initial value problem,"[6] defined by the condition

$$\boldsymbol{\theta}_0(0)=\boldsymbol{\theta}_1(0)=0 \tag{22}$$

represents a "pure Λ" initially. For this case, the integration of (18) yields

$$\boldsymbol{\theta}_1(\tau_0)=-i\int_0^{\tau_0}e^{i(\nabla^2+\mu)\lambda}U\boldsymbol{\chi}_0 d\lambda\,. \tag{23}$$

Employing the notation[7]

$$\zeta(\Omega)\equiv-i\int_0^\infty e^{i\Omega\lambda}d\lambda=\frac{\mathcal{P}}{\Omega}-\pi i\delta(\Omega)$$

and the symbol $\underset{\tau_0}{\sim}$ to read "asymptotically in τ_0," we obtain

$$\boldsymbol{\theta}_1(\tau_0)\underset{\tau_0}{\sim}\zeta(\nabla^2+\mu)U\boldsymbol{\chi}_0 \tag{24}$$

[5] This model is in fact *equivalent* with the lowest sector of the Lee model (Sandri, 1958, unpublished; see Ref. 4).

[6] For the analogous situation in statistical dynamics see Chap. 7, Sec. A of Ref. 1.

[7] See, for example, W. Heitler, *Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd. ed., p. 69.

and its complex conjugate

$$\boldsymbol{\theta}_1^*(\tau_0) \underset{\tau_0}{\sim} -\zeta(-\nabla^2-\mu)U\chi_0^*. \quad (25)$$

Inserting (22) into (20) we find

$$\frac{\partial \mathbf{P}_0}{\partial \tau_1}+\frac{\partial \mathbf{P}_1}{\partial \tau_0}=0, \quad (26)$$

which, by virtue of (17), can be integrated to

$$\mathbf{P}_1(\tau_0)=\mathbf{P}_1(0)-\tau_0\frac{\partial \mathbf{P}_0}{\partial \tau_1}. \quad (27)$$

From the requirement (iii) that we imposed on the extended functions [Eq. (14)] we must have

$$\epsilon \mathbf{P}_1/\mathbf{P}_0=0(\epsilon) \quad \text{for all } \tau_0. \quad (28)$$

From (27) we then conclude

$$\partial \mathbf{P}_0/\partial \tau_1=0 \quad (29)$$

and hence, from (26),

$$\partial \mathbf{P}_1/\partial \tau_0=0. \quad (30)$$

By entirely similar reasoning we obtain from (19), for the simple initial value problem (22),

$$\chi_0(\tau_0,\tau_1,\tau_2)=e^{-i\mu\tau_0}\chi_0(0,0,\tau_2) \quad (31)$$

and

$$\partial \chi_1/\partial \tau_0-\mu\chi_1=0. \quad (32)$$

Turning now to the second-order equation for $\partial \mathbf{P}/\partial t$,

$$i\frac{\partial \mathbf{P}_2}{\partial \tau_0}+i\frac{\partial \mathbf{P}_1}{\partial \tau_1}+i\frac{\partial \mathbf{P}_0}{\partial \tau_2}=2i\,\mathrm{Im}\left[\chi_0^*\int U\boldsymbol{\theta}_1 d\mathbf{x}\right]$$
$$\underset{\tau_0}{\sim} 2i\,\mathrm{Im}\left[\chi_0^*\int U\zeta(\nabla^2+\mu)Ud\mathbf{x}\chi_0\right], \quad (33)$$

we obtain

$$\partial \mathbf{P}_2/\partial \tau_0 \underset{\tau_0}{\sim} 0 \quad (34)$$

$$\partial \mathbf{P}_1/\partial \tau_1=0. \quad (35)$$

Inserting (34) and (35) into (33), we conclude that

$$\partial \mathbf{P}_0/\partial \tau_2=-\lambda \mathbf{P}_0 \quad (36)$$

with the expression for the decay constant[8]

$$\lambda=-2\,\mathrm{Im}\int U\zeta U d\mathbf{x}$$
$$=+2\pi\int U\delta(\nabla^2+\mu)Ud\mathbf{x}. \quad (37)$$

[8] This result can also be derived from the second-order equation for χ. We find in fact

$$i\partial\chi_0/\partial\tau_2=E\chi_0$$

with $E=\int U\zeta(\nabla^2+\mu)Ud\mathbf{x}$. Note that $\lambda=-2\,\mathrm{Im}E$.

We now shall prove that the exponential decay law exhibited for the simple initial value problem [Eq. (36)] results from a general class[9] of initial $\boldsymbol{\theta}(0)$.

Using the integral (21), Eqs. (18), (19), and (20) yield, respectively,

$$i\frac{\partial \boldsymbol{\theta}_1}{\partial \tau_0}+\nabla^2\boldsymbol{\theta}_1=U\chi_0-ie^{i\nabla^2\tau_0}\frac{\partial \boldsymbol{\theta}_0(0)}{\partial \tau_1}, \quad (39)$$

$$i\frac{\partial \chi_1}{\partial \tau_0}-\mu\chi_1=-i\frac{\partial \chi_0}{\partial \tau_1}+\int Ue^{i\nabla^2\tau_0}\boldsymbol{\theta}_0(0)d\mathbf{x}, \quad (40)$$

$$i\frac{\partial \mathbf{P}_1}{\partial \tau_0}+i\frac{\partial \mathbf{P}_0}{\partial \tau_1}=2i\,\mathrm{Im}\left[\chi_0^*\int Ue^{i\nabla^2\tau_0}\boldsymbol{\theta}_0(0)d\mathbf{x}\right]. \quad (41)$$

Since the wave function $\exp[i\nabla^2\tau_0]\boldsymbol{\theta}_0(0)$ represents a free wave packet, we have

$$e^{i\nabla^2\tau_0}\boldsymbol{\theta}_0(0) \underset{\tau_0}{\sim} 0 \quad (42)$$

and therefore, from (39),

$$\boldsymbol{\theta}_1(\tau_0)-e^{i\nabla^2\tau_0}\boldsymbol{\theta}_1(0) \underset{\tau_0}{\sim} \zeta(\nabla^2+\mu)U\chi_0. \quad (43)$$

From (40) and (41) we find that Eqs. (29) to (32) are valid asymptotically for large τ_0 provided[10]

$$\int_0^\infty d\lambda\int Ue^{i(\nabla^2+\mu)\lambda}\boldsymbol{\theta}_0(0)d\mathbf{x}<\infty. \quad (44)$$

From (33) we also find, taking (43) into account, that we shall recover the exponential behavior (34) to (36) if

$$\int_0^\infty d\lambda\int Ue^{i(\nabla^2+\mu)\lambda}\boldsymbol{\theta}_1(0)d\mathbf{x}<\infty. \quad (45)$$

The equations (44) and (45) are the conditions that the wave functions $\boldsymbol{\theta}_0$ and $\boldsymbol{\theta}_1$ must satisfy initially if the quantity $\mathbf{P}_0$, which "forgets" how the system was prepared, is to be defined.

[9] For the analogous situation in statistical dynamics see Chap. 7, Sec. 8 of Ref. 1.

[10] The transient approach to the epxonential regime, that is, the behavior on the fast (τ_0) clock, can be readily calculated with our method. Thus, in first order we obtain from (20), by virtue of (29), the fast rate of change

$$i\partial\mathbf{P}_1/\partial\tau_0=2i\,\mathrm{Im}\left[\chi_0^*\int U\boldsymbol{\theta}_0 d\mathbf{x}\right]$$

and for the second order we obtain, by virtue of (35) and (36),

$$i\partial\mathbf{P}_2/\partial\tau_0=2i\,\mathrm{Im}\left[\chi_0^*\int Ue^{i\nabla^2\tau_0}\boldsymbol{\theta}_1(0)d\mathbf{x}+\chi_1^*\int U\boldsymbol{\theta}_0 d\mathbf{x}\right.$$
$$\left.+i\chi_0^*\int U\int_{\tau_0}^\infty e^{i(\nabla^2+\mu)\lambda}d\lambda U\chi_0 d\mathbf{x}\right].$$